

STABLE DETERMINATION OUTSIDE A CLOAKING REGION OF TWO TIME-DEPENDENT COEFFICIENTS IN AN HYPERBOLIC EQUATION FROM DIRICHLET TO NEUMANN MAP

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ABSTRACT. In this paper, we treat the inverse problem of determining two time-dependent coefficients appearing in a dissipative wave equation, from measured Neumann boundary observations. We establish in dimension $n \geq 2$, stability estimates with respect to the Dirichlet-to-Neumann map of these coefficients provided that are known outside a cloaking regions. Moreover, we prove that it can be stably recovered in larger subsets of the domain by enlarging the set of data.

Keywords: Inverse problems, Dissipative wave equation, Time-dependent coefficients, Stability estimates.

1. INTRODUCTION

1.1. Statement of the problem. This paper deals with the inverse problem of determining two time-dependent coefficients in a dissipative wave equation from boundary observations. Let Ω be a bounded domain of \mathbb{R}^n , $n \geq 2$, with C^∞ boundary $\Gamma = \partial\Omega$. Given $T > 0$, we introduce the following dissipative wave equation

$$(1.1) \quad \begin{cases} \partial_t^2 u - \Delta u + a(x, t) \partial_t u + b(x, t) u = 0 & \text{in } Q = \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x) & \text{in } \Omega, \\ u(x, t) = f(x, t) & \text{on } \Sigma = \Gamma \times (0, T), \end{cases}$$

where $f \in H^1(\Sigma)$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, and the coefficients $a \in C^2(Q)$ and $b \in C^1(Q)$ are assumed to be real valued. It is well known (see [14]) that if $f(\cdot, 0) = u_0|_\Gamma$, there exists a unique solution u to the equation (1.1) satisfying

$$u \in \mathcal{C}([0, T], H^1(\Omega)) \cap \mathcal{C}^1([0, T], L^2(\Omega)).$$

Moreover, there exists $C > 0$, such that

$$(1.2) \quad \|\partial_\nu u\|_{L^2(\Sigma)} + \|u(\cdot, t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot, t)\|_{L^2(\Omega)} \leq C (\|f\|_{H^1(\Sigma)} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}).$$

Here ν denotes the unit outward normal to Γ at x and $\partial_\nu u$ stands for $\nabla u \cdot \nu$.

In the present paper, we address the uniqueness and the stability issues in the study of an inverse problem for the dissipative wave equation (1.1), in the presence of an absorbing coefficient a and a potential b that depend on both space and time variables. We consider three different sets of data and we aim to show that a and b can be recovered in some specific subsets of the domain, by probing it with disturbances generated on the boundary. The Dirichlet data f is considered as a disturbance that is used to probe the medium which is assumed to be quiet initially.

The problem of identifying coefficients appearing in hyperbolic boundary value problems was treated very well and there are many works that are relevant to this topic. In the case where the unknown coefficient

is depending only on the spatial variable, Rakesh and Symes [22] proved by means of geometric optics solutions, a uniqueness result in recovering a time-independent potential in a wave equation from global Neumann data. The uniqueness by local Neumann data, was considered by Eskin [13] and Isakov [16]. In [5], Bellassoued, Choulli and Yamamoto proved a log-type stability estimate, in the case where the Neumann data are observed on any arbitrary subset of the boundary. Isakov and Sun [18] proved that the knowledge of local Dirichlet-to-Neumann map yields a stability result of Hölder type in determining a coefficient in a subdomain. As for the stability obtained from global Neumann data, one can see Sun [29], Cipolatti and Lopez [11]. The case of Riemannian manifold was considered by Bellassoued and Dos Santos Ferreira [6], Stefanov and Uhlmann [28].

All the mentioned papers are concerned only with time-independent coefficients. In the case where the coefficient is also depending on the time variable, There is a uniqueness result proved by Ramm and Rakesh [23], in which they showed that a time-dependent coefficient appearing in a wave equation with zero initial conditions, can be uniquely determined from the knowledge of global Neumann data, but only in a precise subset of the cylindrical domain Q that is made of lines making an angle of 45° with the t -axis and meeting the planes $t = 0$ and $t = T$ outside \overline{Q} . However, inspired by the work of [30], Isakov proved in [17], that the time-dependent coefficient can be recovered from the responses of the medium for all possible initial data, over the whole domain Q .

It is clear that with zero initial data, there is no hope to recover a time-dependent coefficient appearing in a hyperbolic equation over the whole cylindrical domain, even from the knowledge of global Neumann data, because the value of the solution can be effected by the value of the initial conditions, which is actually due to a fundamental concept concerning hyperbolic equations called the domain of dependence (see [14]). Moreover, we can prove that the backward light-cone with base Ω is a cloaking region, that is we can not uniquely recover the coefficients in this region.

As for uniqueness results, we have also the paper of Stefanov [26], in which he proved that a time-dependent potential appearing in a wave equation can be uniquely recovered from scattering data and the paper of Ramm and Sjöstrand [24], in which they proved a uniqueness result on an infinite time-space cylindrical domain $\Omega \times \mathbb{R}_t$.

The stability in this case, was considered by Salazar [25], who extended the result of Ramm and Sjöstrand [24] to more general coefficients and he established a stability result for compactly supported coefficients provided T is sufficiently large. We also refer to the works of Kian [19, 20] who followed techniques used by Bellassoued, Jellali and Yamamoto [7, 8] and proved uniqueness and a log-log type stability estimate from the knowledge of partial Neumann data. As for stability results from global Neumann data we refer to Ben Aïcha [10] who proved recently a stability of log-type in recovering a zeroth order time-dependent coefficient in different regions of the cylindrical domain by considering different sets of data. We also refer to Waters [32] who derived, in Riemannian case, conditional Hölder stability estimates for the X-ray transform of the time-dependent potential appearing in the wave equation.

As for results of hyperbolic inverse problems dealing with single measurement data, one can see [2, 3, 9, 12, 15, 27] and the references therein.

Inspired by the work of Bellassoued [4] and following the same strategy as in Ben Aïcha [10], we prove in this paper stability estimates in the recovery of the unknown coefficients a and b via different types of measurements and over different subsets of the domain Q .

1.2. Main results. Before stating our main results we first introduce the following notations.

Let $r > 0$ be such that $T > 2r$ and $\Omega \subseteq B(0, r/2) = \{x \in \mathbb{R}^n, |x| < r/2\}$. We set $Q_r = B(0, r/2) \times (0, T)$ and we consider the annular region around the domain Ω ,

$$\mathcal{A}_r = \left\{x \in \mathbb{R}^n, \frac{r}{2} < |x| < T - \frac{r}{2}\right\},$$

and the forward and backward light cone:

$$\begin{aligned} \mathcal{C}_r^+ &= \left\{(x, t) \in Q_r, |x| < t - \frac{r}{2}, t > \frac{r}{2}\right\}, \\ \mathcal{C}_r^- &= \left\{(x, t) \in Q_r, |x| < T - \frac{r}{2} - t, T - \frac{r}{2} > t\right\}. \end{aligned}$$

Finally, we denote

$$Q_r^* = \mathcal{C}_r^+ \cap \mathcal{C}_r^-, \quad Q_{r,*} = Q \cap Q_r^*, \quad \text{and } quad Q_{r,\sharp} = Q \cap \mathcal{C}_r^+.$$

We remark that the open subset $Q_{r,*}$ is made of lines making an angle of 45° with the t -axis and meeting the planes $t = 0$ and $t = T$ outside $\overline{Q_r}$ and $Q_{r,\sharp}$ is made of lines making an angle of 45° with the t -axis and meeting only the planes $t = 0$ outside $\overline{Q_r}$. We notice that $Q_{r,*} \subset Q_{r,\sharp} \subset Q$. Note, that in the particular case where $\Omega = B(0, r/2)$, we have $Q_{r,*} = Q_r^*$ and $Q_{r,\sharp} = \mathcal{C}_r^+$ (see Figure 1 in [10]).

In the present paper, we will prove stability estimates for the inverse problem under consideration in three cases. We will consider three different sets of data and we will prove that the coefficients a and b can be stably determined in three different regions of the cylindrical domain Q . Given $M_1, M_2 > 0$, we consider the set of admissible coefficients a and b :

$$\mathcal{A}(M_1, M_2) = \{(a, b) \in \mathcal{C}^2(\overline{Q_r}) \times \mathcal{C}^1(\overline{Q_r}); \|a\|_{C^2(Q)} \leq M_1, \|b\|_{C^1(Q)} \leq M_2\}.$$

Finally, we define the following space

$$\mathcal{H}_0^1 = \left\{f \in H^1(\Sigma), f(\cdot, 0) = 0 \text{ on } \Gamma\right\},$$

equipped with the norm of $H^1(\Sigma)$.

1.2.1. Determination of coefficients from boundary measurements: In the first case, we will assume that the initial conditions (u_0, u_1) are zero and our set of data will be given only by boundary measurements enclosed by the Dirichlet-to-Neumann map $\Lambda_{a,b}$ defined as follows

$$\begin{aligned} \Lambda_{a,b} : \mathcal{H}_0^1(\Sigma) &\longrightarrow L^2(\Sigma) \\ f &\longmapsto \partial_\nu u. \end{aligned}$$

Note that from (1.2) we have $\Lambda_{a,b}$ is continuous from $\mathcal{H}_0^1(\Sigma)$ to $L^2(\Sigma)$. We denote by $\|\Lambda_{a,b}\|$ its norm in $\mathcal{L}(\mathcal{H}_0^1(\Sigma), L^2(\Sigma))$. We can prove that it is hopeless to uniquely determine a and b in the case where these coefficients are supported in the cloaking region

$$\mathcal{C} = \left\{(x, t) \in Q_r; |x| < \frac{r}{2} - t, t < \frac{r}{2}\right\}.$$

The first result of this paper can be stated as follows

Theorem 1.1. *Let $T > 2\text{Diam}(\Omega)$. There exist constants $C > 0$ and $\mu_1, \mu_2 \in (0, 1)$ such that we have*

$$\|a_2 - a_1\|_{L^\infty(Q_{r,*})} \leq C \left(\|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\|^{\mu_1} + |\log \|\Lambda_{a_1,b_1} - \Lambda_{a_2,b_2}\||^{-1} \right)^{\mu_2},$$

for any $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $\|a_i\|_{H^p(Q)} \leq M_1$, for some $p > n/2 + 3/2$, $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q_r} \setminus Q_{r,*}$ and $(\nabla a_1, \nabla b_1) = (\nabla a_2, \nabla b_2)$ on $\partial Q_r \cap \partial Q_{r,*}$. Here C depends only on Ω, M_1, M_2, T and n .

The above statement claims stable determination of the time-dependent coefficient a from the Neumann boundary measurements $\Lambda_{a,b}$ in $Q_{r,*} \subset Q$, provided a is known outside $Q_{r,*}$. By Theorem 1.1, we can readily derive the following result

Theorem 1.2. *Let $T > 2\text{Diam}(\Omega)$. There exist two constants $C > 0$ and $\mu \in (0, 1)$ such that we have*

$$\|b_2 - b_1\|_{H^{-1}(Q_{r,*})} \leq C \left(\log |\log \|\Lambda_{a_2,b_2} - \Lambda_{a_1,b_1}\|^\mu| \right)^{-1},$$

for any $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $\|a_i\|_{H^p(Q)} \leq M_1$, for some $p > n/2 + 3/2$, $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q_r} \setminus Q_{r,*}$ and $(\nabla a_1, \nabla b_1) = (\nabla a_2, \nabla b_2)$ on $\partial Q_r \cap \partial Q_{r,*}$. Here C depends only on Ω , M_1 , M_2 , T and n .

This mentioned result shows that the time-dependent potential b can also be stably determined, from the knowledge of the boundary measurements $\Lambda_{a,b}$ in the same subset $Q_{r,*} \subset Q$, provided it is known outside this region. As a consequence, we have the following existence result

Corollary 1.3. *Under the same assumptions of Theorem 1.1 and Theorem 1.2, we have that $\Lambda_{a_1,b_1} = \Lambda_{a_2,b_2}$ implies $a_1 = a_2$ and $b_1 = b_2$ in $Q_{r,*}$.*

1.2.2. Determination of coefficients from boundary measurements and final data: In order to extend the above results to a larger region $Q_{r,\sharp} \supset Q_{r,*}$, we require more information about the solution u of the wave equation 1.1. So, in this case we will add the final data of the solution u . This leads to defining the following boundary operator (response operator):

$$\begin{aligned} \mathcal{R}_{a,b} : \mathcal{H}_0^1(\Sigma) &\longrightarrow \mathcal{K} := L^2(\Sigma) \times H^1(\Omega) \times L^2(\Omega) \\ f &\longmapsto (\partial_\nu u, u(\cdot, T), \partial_t u(\cdot, T)). \end{aligned}$$

We conclude from (1.2), that $\mathcal{R}_{a,b}$ is a continuous operator from $\mathcal{H}_0^1(\Sigma)$ to \mathcal{K} . We denote by $\|\mathcal{R}_{a,b}\|$ its norm in $\mathcal{L}(\mathcal{H}_0^1(\Sigma), \mathcal{K})$.

Theorem 1.4. *Let $T > 2\text{Diam}(\Omega)$. There exist constants $C > 0$ and $\mu_1, \mu_2 \in (0, 1)$ such that we have*

$$\|a_2 - a_1\|_{L^\infty(Q_{r,\sharp})} \leq C \left(\|\mathcal{R}_{a_1,b_1} - \mathcal{R}_{a_2,b_2}\|^{\mu_1} + |\log \|\mathcal{R}_{a_1,b_1} - \mathcal{R}_{a_2,b_2}\|^{-1}|^{\mu_2} \right),$$

for any $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $\|a_i\|_{H^p(Q)} \leq M_1$, for some $p > n/2 + 3/2$, $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q_r} \setminus Q_{r,\sharp}$ and $(\nabla a_1, \nabla b_1) = (\nabla a_2, \nabla b_2)$ on $\partial Q_r \cap \partial Q_{r,\sharp}$. Here C depends only on Ω , M_1 , M_2 , T and n .

From the above statement, we can readily derive the following consequence

Theorem 1.5. *Let $T > 2\text{Diam}(\Omega)$. There exist two constants $C > 0$ and $\mu \in (0, 1)$ such that we have*

$$\|b_2 - b_1\|_{H^{-1}(Q_{r,\sharp})} \leq C \left(\log |\log \|\mathcal{R}_{a_2,b_2} - \mathcal{R}_{a_1,b_1}\|^\mu| \right)^{-1},$$

for any $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $\|a_i\|_{H^p(Q)} \leq M_1$, for some $p > n/2 + 3/2$, $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q_r} \setminus Q_{r,\sharp}$ and $(\nabla a_1, \nabla b_1) = (\nabla a_2, \nabla b_2)$ on $\partial Q_r \cap \partial Q_{r,\sharp}$. Here C depends only on Ω , M_1 , M_2 , T and n .

As a consequence, we have the following uniqueness result

Corollary 1.6. *Under the same assumptions of Theorem 1.4 and Theorem 1.5, we have that $\mathcal{R}_{a_1,b_1} = \mathcal{R}_{a_2,b_2}$ implies $a_1 = a_2$ and $b_1 = b_2$ in $Q_{r,\sharp}$.*

1.2.3. *Determination of coefficients from boundary measurements and final data by varying the initial data:* In the first and the second case, we can see that there is no hope to recover the unknown coefficients a and b over the whole domain, since the initial data (u_0, u_1) are zero. However, we shall prove that this is no longer the case by considering all possible initial data.

We define the following space $\mathcal{F} = H^1(\Sigma) \times H^1(\Omega) \times L^2(\Omega)$. In this case we will consider observations given by the following operator:

$$\begin{aligned} \mathcal{I}_{a,b} : \quad \mathcal{F} &\longrightarrow \mathcal{K} \\ (f, u_0, u_1) &\longmapsto (\partial_\nu u, u(\cdot, T), \partial_t u(\cdot, T)). \end{aligned}$$

By (1.2), we deduce that $\mathcal{I}_{a,b}$ is continuous from \mathcal{F} into \mathcal{K} , we denote by $\|\mathcal{I}_{a,b}\|$ its norm in $\mathcal{L}(\mathcal{F}, \mathcal{K})$. Having said that, we are now in position to state the last main result.

Theorem 1.7. *There exist constants $C > 0$ and $\mu_1, \mu_2 \in (0, 1)$ such that the following estimate holds*

$$\|a_2 - a_1\|_{L^\infty(Q)} \leq C \left(\|\mathcal{I}_{a_1, b_1} - \mathcal{I}_{a_2, b_2}\|^{\mu_1} + |\log \|\mathcal{I}_{a_1, b_1} - \mathcal{I}_{a_2, b_2}\||^{-1} \right)^{\mu_2},$$

for any $(a_i, b_i) \in \mathcal{C}^2(\overline{Q}) \times \mathcal{C}^1(\overline{Q})$, such that $\|a_i\|_{\mathcal{C}^2(Q)} + \|a_i\|_{H^p(Q)} \leq M_1$ for some $p > n/2 + 3/2$, $\|b_i\|_{\mathcal{C}^1(Q)} \leq M_2$ and $(\nabla a_1, \nabla b_1) = (\nabla a_2, \nabla b_2)$ on Σ . Here C depends only on Ω, M_1, M_2, T and n .

As a consequence, we have the following result

Theorem 1.8. *There exist two constants $C > 0$ and $\mu \in (0, 1)$ such that the following estimate holds*

$$\|b_2 - b_1\|_{H^{-1}(Q)} \leq C \left(\log |\log \|\mathcal{I}_{a_2, b_2} - \mathcal{I}_{a_1, b_1}\||^\mu \right)^{-1},$$

for any $(a_i, b_i) \in \mathcal{C}^2(\overline{Q}) \times \mathcal{C}^1(\overline{Q})$, such that $\|a_i\|_{\mathcal{C}^2(Q)} + \|a_i\|_{H^p(Q)} \leq M_1$ for some $p > n/2 + 3/2$, $\|b_i\|_{\mathcal{C}^1(Q)} \leq M_2$ and $(\nabla a_1, \nabla b_1) = (\nabla a_2, \nabla b_2)$ on Σ . Here C depends only on Ω, M_1, M_2, T and n .

Corollary 1.9. *Under the same assumptions of Theorem 1.7 and Theorem 1.8, we have that $\mathcal{I}_{a_1, b_1} = \mathcal{I}_{a_2, b_2}$ implies that $a_1 = a_2$ and $b_1 = b_2$ everywhere in Q .*

The outline of this paper is as follows. Section 2 is devoted to the construction of geometric optics solutions to the equation (1.1). Using these particular solutions, we establish in Section 3 stability estimates for the absorbing coefficient a and the potential b . Section 4 and 5 are devoted to the proof of the results of the second and third case respectively. In appendix A, we develop the proof of an analytic technical result.

2. CONSTRUCTION OF GEOMETRIC OPTICS SOLUTIONS

The present section is devoted to the construction of suitable geometrical optics solutions for the dissipative wave equation (1.1), which are key ingredients to the proof of our main results. The construction here is a modification of a similar result in [10]. We shall first state the following lemma which is needed to prove the main statement of this section.

Lemma 2.1. *(see [14]) Let $T, M_1, M_2 > 0$, $a \in L^\infty(Q)$ and $b \in L^\infty(Q)$, such that $\|a\|_{L^\infty(Q)} \leq M_1$ and $\|b\|_{L^\infty(Q)} \leq M_2$. Assume that $F \in L^1(0, T; L^2(\Omega))$. Then, there exists a unique solution u to the following equation*

$$(2.3) \quad \begin{cases} \partial_t^2 u - \Delta u + a(x, t) \partial_t u + b(x, t) u(x, t) = F(x, t) & \text{in } Q, \\ u(x, 0) = 0 = \partial_t u(x, 0) & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \Sigma, \end{cases}$$

such that

$$u \in \mathcal{C}([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)),$$

Moreover, there exists a constant $C > 0$ such that

$$(2.4) \quad \|\partial_t u(\cdot, t)\|_{L^2(\Omega)} + \|\nabla u(\cdot, t)\|_{L^2(\Omega)} \leq C \|F\|_{L^1(0, T; L^2(\Omega))}.$$

Armed with the above lemma, we may now construct suitable geometrical optics solutions to the dissipative wave equation (1.1) and to its retrograde problem. For this purpose, we consider $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and notice that for all $\omega \in \mathbb{S}^{n-1}$ the function ϕ given by

$$(2.5) \quad \phi(x, t) = \varphi(x + t\omega),$$

solves the following transport equation

$$(2.6) \quad (\partial_t - \omega \cdot \nabla)\phi(x, t) = 0.$$

We are now in position to prove the following statement

Lemma 2.2. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. Given $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we consider the function ϕ defined by (2.5). Then, for any $\lambda > 0$, the following equation*

$$(2.7) \quad \partial_t^2 u - \Delta u + a(x, t)\partial_t u + b(x, t)u = 0 \quad \text{in } Q,$$

admits a unique solution

$$u^+ \in \mathcal{C}([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)),$$

of the following form

$$(2.8) \quad u^+(x, t) = \phi(x, t)A^+(x, t)e^{i\lambda(x \cdot \omega + t)} + r_\lambda^+(x, t),$$

where $A^+(x, t)$ is given by

$$(2.9) \quad A^+(x, t) = \exp\left(-\frac{1}{2} \int_0^t a(x + (t-s)\omega, s) ds\right),$$

and $r_\lambda^+(x, t)$ satisfies

$$(2.10) \quad r_\lambda^+(x, 0) = \partial_t r_\lambda^+(x, 0) = 0, \quad \text{in } \Omega, \quad r_\lambda^+(x, t) = 0 \quad \text{on } \Sigma.$$

Moreover, there exists a positive constant $C > 0$ such that

$$(2.11) \quad \lambda \|r_\lambda^+(\cdot, t)\|_{L^2(\Omega)} + \|\partial_t r_\lambda^+(\cdot, t)\|_{L^2(\Omega)} \leq C \|\varphi\|_{H^3(\mathbb{R}^n)}.$$

Proof. We proceed as in the proof of a similar result in [10]. We put

$$g(x, t) = -\left(\partial_t^2 - \Delta + a(x, t)\partial_t + b(x, t)\right)\left(\phi(x, t)A^+(x, t)e^{i\lambda(x \cdot \omega + t)}\right).$$

In light of (2.7) and (2.8), it will be enough to prove the existence of $r^+ = r_\lambda^+$ satisfying

$$(2.12) \quad \begin{cases} \left(\partial_t^2 - \Delta + a(x, t)\partial_t + b(x, t)\right)r^+ = g(x, t), \\ r^+(x, 0) = \partial_t r^+(x, 0) = 0, \\ r^+(x, t) = 0, \end{cases}$$

and obeying the estimate (2.11). From (2.6) and using the fact that $A^+(x, t)$ solves the following equation

$$2(\partial_t - 2\omega \cdot \nabla)A^+(x, t) = -a(x, t)A^+(x, t),$$

we obtain the following identity

$$g(x, t) = -e^{i\lambda(x \cdot \omega + t)} \left(\partial_t^2 - \Delta + a(x, t) \partial_t + b(x, t) \right) \left(\phi(x, t) A^+(x, t) \right) = -e^{i\lambda(x \cdot \omega + t)} g_0(x, t),$$

where $g_0 \in L^1(0, T, L^2(\Omega))$. Thus, in view of Lemma 2.1, there exists a unique solution

$$r^+ \in \mathcal{C}([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)),$$

satisfying (2.12). Let us now define by w the following function

$$(2.13) \quad w(x, t) = \int_0^t r^+(x, s) ds.$$

We integrate the equation (2.12) over $[0, t]$, for $t \in (0, T)$. Then, in view of (2.13), we have

$$\begin{aligned} \left(\partial_t^2 - \Delta + a(x, t) \partial_t + b(x, t) \right) w(x, t) &= \int_0^t g(x, s) ds + \int_0^t \left(b(x, t) - b(x, s) \right) r^+(x, s) ds \\ &\quad + \int_0^t \partial_s a(x, s) r^+(x, s) ds. \end{aligned}$$

Therefore, w is a solution to the following equation

$$\begin{cases} \left(\partial_t^2 - \Delta + a(x, t) \partial_t + b(x, t) \right) w(x, t) = F_1(x, t) + F_2(x, t) & \text{in } Q, \\ w(x, 0) = 0 = \partial_t w(x, 0) & \text{in } \Omega, \\ w(x, t) = 0 & \text{on } \Sigma, \end{cases}$$

where F_1 and F_2 are given by

$$(2.14) \quad F_1(x, t) = \int_0^t g(x, s) ds,$$

and

$$F_2(x, t) = \int_0^t \left(b(x, t) - b(x, s) \right) r^+(x, s) ds + \int_0^t \partial_s a(x, s) r^+(x, s) ds.$$

Let $\tau \in [0, T]$. Applying Lemma 2.1 on the interval $[0, \tau]$, we get

$$\|\partial_t w(\cdot, \tau)\|_{L^2(\Omega)}^2 \leq C \left(\|F_1\|_{L^2(0, T; L^2(\Omega))}^2 + T (M_1^2 + 4M_2^2) \int_0^\tau \int_\Omega \int_0^t |r^+(x, s)|^2 ds dx dt \right).$$

From (2.13), we get

$$\begin{aligned} \|\partial_t w(\cdot, \tau)\|_{L^2(\Omega)}^2 &\leq C \left(\|F_1\|_{L^2(0, T; L^2(\Omega))}^2 + \int_0^\tau \int_0^t \|\partial_s w(\cdot, s)\|_{L^2(\Omega)}^2 ds dt \right) \\ &\leq C \left(\|F_1\|_{L^2(0, T; L^2(\Omega))}^2 + T \int_0^\tau \|\partial_s w(\cdot, s)\|_{L^2(\Omega)}^2 ds \right). \end{aligned}$$

Therefore, from Gronwall's Lemma, we find out that

$$\|\partial_t w(\cdot, \tau)\|_{L^2(\Omega)}^2 \leq C \|F_1\|_{L^2(0, T; L^2(\Omega))}^2.$$

As a consequence, in light of (2.13), we conclude that $\|r^+(\cdot, t)\|_{L^2(\Omega)} \leq C \|F_1\|_{L^2(0, T; L^2(\Omega))}$. Further, according to (2.14), F_1 can be written as follows

$$F_1(x, t) = \int_0^t g(x, s) ds = \frac{1}{i\lambda} \int_0^t g_0(x, s) \partial_s (e^{i\lambda(x \cdot \omega + s)}) ds.$$

Integrating by parts with respect to s , we conclude that there exists a positive constant $C > 0$ such that

$$\|r^+(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{\lambda} \|\varphi\|_{H^3(\mathbb{R}^n)}.$$

Finally, since $\|g\|_{L^2(0,T;L^2(\Omega))} \leq C\|\varphi\|_{H^3(\mathbb{R}^n)}$, the energy estimate (2.4) associated to the problem (2.12) yields

$$\|\partial_t r^+(\cdot, t)\|_{L^2(\Omega)} + \|\nabla r^+(\cdot, t)\|_{L^2(\Omega)} \leq C\|\varphi\|_{H^3(\mathbb{R}^n)}.$$

This completes the proof of the lemma. \square

As a consequence we have the following lemma

Lemma 2.3. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. Given $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, we consider the function ϕ defined by (2.5). Then, the following equation*

$$(2.15) \quad \partial_t^2 u - \Delta u - a(x, t)\partial_t u + b(x, t)u = 0 \quad \text{in } Q,$$

admits a unique solution

$$u^- \in \mathcal{C}([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)),$$

of the following form

$$(2.16) \quad u^-(x, t) = \varphi(x + t\omega)A^-(x, t)e^{-i\lambda(x \cdot \omega + t)} + r_\lambda^-(x, t),$$

where $A^-(x, t)$ is given by

$$(2.17) \quad A^-(x, t) = \exp\left(\frac{1}{2} \int_0^t a(x + (t-s)\omega, s) ds\right),$$

and $r_\lambda^-(x, t)$ satisfies

$$(2.18) \quad r_\lambda^-(x, T) = \partial_t r_\lambda^-(x, T) = 0, \quad \text{in } \Omega, \quad r_\lambda^-(x, t) = 0 \quad \text{on } \Sigma.$$

Moreover, there exists a constant $C > 0$ such that

$$(2.19) \quad \lambda\|r_\lambda^-(\cdot, t)\|_{L^2(\Omega)} + \|\partial_t r_\lambda^-(\cdot, t)\|_{L^2(\Omega)} \leq C\|\varphi\|_{H^3(\mathbb{R}^n)}.$$

Proof. We prove this result by proceeding as in the proof of Lemma 2.2. Putting

$$\tilde{g}(x, t) = -\left(\partial_t^2 - \Delta - a(x, t)\partial_t + b(x, t)\right)\left(\phi(x, t)A^-(x, t)e^{-i\lambda(x \cdot \omega + t)}\right).$$

Then, it would be enough to see that if $r^- = r_\lambda^-$ is solution to the following system

$$\begin{cases} \left(\partial_t^2 - \Delta - a(x, t)\partial_t + b(x, t)\right)r^-(x, t) = \tilde{g}(x, t) & \text{in } Q, \\ r^-(x, T) = 0 = \partial_t r^-(x, T) & \text{in } \Omega, \\ r^-(x, t) = 0 & \text{on } \Sigma, \end{cases}$$

then, $r^+(x, t) = r^-(x, T - t)$ is a solution to (2.12) with $g(x, t) = \tilde{g}(x, T - t)$ and $a(x, t), b(x, t)$ are replaced by $a(x, T - t)$ and $b(x, T - t)$. \square

3. DETERMINATION OF COEFFICIENTS FROM BOUNDARY MEASUREMENTS

In this section we prove stability estimates for the absorbing coefficient a and the potential b appearing in the initial boundary value problem (1.1) by the use of the geometrical optics solutions constructed in section 2 and the light-ray transform. We assume that $\text{Supp } \varphi \subset \mathcal{A}_r$, in such a way we have

$$\text{Supp } \varphi \cap \Omega = \emptyset \quad \text{and} \quad (\text{Supp } \varphi \pm T\omega) \cap \Omega = \emptyset, \quad \forall \omega \in \mathbb{S}^{n-1}.$$

3.1. Stability for the absorbing coefficient. The present section is devoted to the proof of Theorem 1.1. Our goal here is to show that the time dependent coefficient a depends stably on the Dirichlet-to-Neumann map $\Lambda_{a,b}$. In the rest of this section, we define a in \mathbb{R}^{n+1} by $a = a_2 - a_1$ in \overline{Q}_r and $a = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q}_r$. We start by collecting a preliminary estimate which is needed to prove the main statement of this section.

3.1.1. Preliminary estimate. The main purpose of this section is to give a preliminary estimate, which relates the difference of the absorbing coefficients to the Dirichlet-to-Neumann map. Let $\omega \in \mathbb{S}^{n-1}$, and $(a_i, b_i) \in \mathcal{A}$ such that $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q}_r \setminus Q_{r,*}$. We set

$$a = a_2 - a_1, \quad b = b_2 - b_1, \quad \text{and} \quad A(x, t) = (A^- A^+)(x, t) = \exp \left(-\frac{1}{2} \int_0^t a(x + (t-s)\omega, s) ds \right).$$

Here, we recall the definition of A^- and A^+

$$A^-(x, t) = \exp \left(\frac{1}{2} \int_0^t a_1(x + (t-s)\omega, s) ds \right), \quad A^+(x, t) = \exp \left(-\frac{1}{2} \int_0^t a_2(x + (t-s)\omega, s) ds \right).$$

The main result of this section can be stated as follows

Lemma 3.1. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exists $C > 0$ such that for any $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathcal{A}_r)$, the following estimate holds true*

$$\left| \int_{\mathbb{R}^n} \varphi^2(y) \left[\exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right] dy \right| \leq C \left(\lambda^2 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2,$$

for any sufficiently large $\lambda > 0$. Here C depends only on Ω , T , M_1 and M_2 .

Proof. In view of Lemma 2.2, and using the fact that $\text{Supp } \varphi \cap \Omega = \emptyset$, there exists a geometrical optics solution u^+ to the equation

$$\begin{cases} \partial_t^2 u^+ - \Delta u^+ + a_2(x, t) \partial_t u^+ + b_2(x, t) u^+ = 0 & \text{in } Q, \\ u^+(x, 0) = \partial_t u^+(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

in the following form

$$(3.20) \quad u^+(x, t) = \varphi(x + t\omega) A^+(x, t) e^{i\lambda(x \cdot \omega + t)} + r_\lambda^+(x, t),$$

corresponding to the coefficients a_2 and b_2 , where $r^+(x, t)$ satisfies (2.10), (2.11). Next, let us denote by f_λ the function

$$f_\lambda(x, t) = u^+(x, t)|_\Sigma = \varphi(x + t\omega) A^+(x, t) e^{i\lambda(x \cdot \omega + t)}.$$

We denote by u_1 the solution of

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + a_1(x, t) \partial_t u_1 + b_1(x, t) u_1 = 0 & \text{in } Q, \\ u_1(x, 0) = \partial_t u_1(x, 0) = 0 & \text{in } \Omega, \\ u_1 = f_\lambda & \text{on } \Sigma. \end{cases}$$

Putting $u = u_1 - u^+$. Then, u is a solution to the following system

$$(3.21) \quad \begin{cases} \partial_t^2 u - \Delta u + a_1(x, t) \partial_t u + b_1(x, t) u = a(x, t) \partial_t u^+ + b(x, t) u^+ & \text{in } Q, \\ u(x, 0) = \partial_t u(x, 0) = 0 & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \Sigma. \end{cases}$$

where $a = a_2 - a_1$ and $b = b_2 - b_1$. On the other hand Lemma 2.3 and the fact that $(\text{Supp } \varphi \pm T\omega) \cap \Omega = \emptyset$, guarantee the existence of a geometrical optic solution u^- to the backward problem of (1.1)

$$\begin{cases} \partial_t^2 u^- - \Delta u^- - a_1(x, t) \partial_t u^- + (b_1(x, t) - \partial_t a_1(x, t)) u^- = 0 & \text{in } Q, \\ u^-(x, T) = 0 = \partial_t u^-(x, T) & \text{in } \Omega, \end{cases}$$

corresponding to the coefficients a_1 and $(-\partial_t a_1 + b_1)$, in the form

$$(3.22) \quad u^-(x, t) = \varphi(x + t\omega) e^{-i\lambda(x \cdot \omega + t)} A^-(x, t) + r_\lambda^-(x, t),$$

where $r_\lambda^-(x, t)$ satisfies (2.18), (2.19). Multiplying the first equation of (3.21) by u^- , integrating by parts and using Green's formula, we obtain

$$(3.23) \quad \int_0^T \int_\Omega a(x, t) \partial_t u^+ u^- dx dt + \int_0^T \int_\Omega b(x, t) u^+ u^- dx dt = \int_0^T \int_\Gamma (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt.$$

On the other hand, by replacing u^+ and u^- by their expressions, we have

$$\begin{aligned} \int_0^T \int_\Omega a(x, t) \partial_t u^+ u^- dx dt &= \int_0^T \int_\Omega a(x, t) \partial_t \varphi(x + t\omega) e^{i\lambda(x \cdot \omega + t)} A^+ r_\lambda^- dx dt \\ &+ \int_0^T \int_\Omega a(x, t) \varphi(x + t\omega) e^{i\lambda(x \cdot \omega + t)} \partial_t A^+ r_\lambda^- dx dt + \int_0^T \int_\Omega a(x, t) \partial_t \varphi(x + t\omega) \varphi(x + t\omega) (A^+ A^-) dx dt \\ &+ \int_0^T \int_\Omega a(x, t) \varphi^2(x + t\omega) \partial_t A^+ A^- dx dt + i\lambda \int_0^T \int_\Omega a(x, t) \varphi(x + t\omega) e^{i\lambda(x \cdot \omega + t)} A^+ r_\lambda^- dx dt \\ &+ \int_0^T \int_\Omega a(x, t) \varphi(x + t\omega) e^{-i\lambda(x \cdot \omega + t)} A^- \partial_t r_\lambda^+ dx dt + i\lambda \int_0^T \int_\Omega a(x, t) \varphi^2(x + t\omega) (A^+ A^-) dx dt \\ &+ \int_0^T \int_\Omega a(x, t) \partial_t r_\lambda^+ r_\lambda^- dx dt = i\lambda \int_0^T \int_\Omega a(x, t) \varphi^2(x + t\omega) A dx dt + \mathcal{I}_\lambda, \end{aligned}$$

where $A = A^+ A^-$. In light of (3.23), we have

$$(3.24) \quad i\lambda \int_0^T \int_\Omega a(x, t) \varphi^2(x + t\omega) A(x, t) dx dt = \int_0^T \int_\Gamma (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt - \int_0^T \int_\Omega b(x, t) u^+ u^- dx dt - \mathcal{I}_\lambda.$$

Note that for λ sufficiently large, we have

$$(3.25) \quad \|u^+ u^-\|_{L^1(Q)} \leq C \|\varphi\|_{H^3(\mathbb{R}^n)}^2, \quad \text{and} \quad |\mathcal{I}_\lambda| \leq C \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

On the other hand, since on Σ , we have $u^+ = f_\lambda$ and $r_\lambda^- = r_\lambda^+ = 0$, then, we get the following estimate

$$\begin{aligned} \left| \int_0^T \int_\Gamma (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt \right| &\leq \| \Lambda_{a_2, b_2} - \Lambda_{a_1, b_1} \| \|f_\lambda\|_{H^1(\Sigma)} \|u^-\|_{L^2(\Sigma)} \\ &\leq \| \Lambda_{a_2, b_2} - \Lambda_{a_1, b_1} \| \|u^+ - r_\lambda^+\|_{H^2(Q)} \|u^- - r_\lambda^-\|_{H^1(Q)} \\ (3.26) \quad &\leq C \lambda^3 \| \Lambda_{a_2, b_2} - \Lambda_{a_1, b_1} \| \|\varphi\|_{H^3(\mathbb{R}^n)}^2. \end{aligned}$$

Consequently, by (3.24), (3.25) and (3.26), we obtain

$$\left| \int_0^T \int_{\Omega} a(x, t) \varphi^2(x + t\omega) A(x, t) dx dt \right| \leq C \left(\lambda^2 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2,$$

where $A(x, t) = \exp \left(-\frac{1}{2} \int_0^t a(x + (t-s)\omega, s) ds \right)$. Then, using the fact $a(x, t) = 0$ outside $Q_{r,*}$ and making this change of variables $y = x + t\omega$, one gets the following estimation

$$\left| \int_0^T \int_{\mathbb{R}^n} a(y - t\omega, t) \varphi^2(y) \exp \left(-\frac{1}{2} \int_0^t a(y - s\omega, s) ds \right) dy dt \right| \leq C \left(\lambda^2 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

Bearing in mind that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} a(y - t\omega, t) \varphi^2(y) \exp \left(-\frac{1}{2} \int_0^t a(y - s\omega, s) ds \right) dy dt \\ = -2 \int_0^T \int_{\mathbb{R}^n} \varphi^2(y) \frac{d}{dt} \left[\exp \left(-\frac{1}{2} \int_0^t a(y - s\omega, s) ds \right) \right] dy dt \\ = -2 \int_{\mathbb{R}^n} \varphi^2(y) \left[\exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right] dy, \end{aligned}$$

we conclude the desired estimate given by

$$\left| \int_{\mathbb{R}^n} \varphi^2(y) \left[\exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right] dy \right| \leq C \left(\lambda^2 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

This completes the proof of the lemma. \square

3.1.2. Stability for the light-ray transform. The light-ray transform \mathcal{R} maps a function $f \in L^1(\mathbb{R}^{n+1})$ defined in \mathbb{R}^{n+1} into the set of its line integrals. More precisely, if $\omega \in \mathbb{S}^{n-1}$ and $(x, t) \in \mathbb{R}^{n+1}$, the function

$$\mathcal{R}(f)(x, \omega) := \int_{\mathbb{R}} f(x - t\omega, t) dt,$$

is the integral of f over the lines $\{(x - t\omega, t), t \in \mathbb{R}\}$. The goal in this section is to obtain an estimate that links the light-ray transform of the absorbing coefficient $a = a_2 - a_1$ to the measurement $\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}$ on a precise set. Using the above lemma, we can control the light-ray transform of a as follows:

Lemma 3.2. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exist $C > 0$, $\delta > 0$, $\beta > 0$ and $\lambda_0 > 0$ such that for all $\omega \in \mathbb{S}^{n-1}$, we have*

$$|\mathcal{R}(a)(y, \omega)| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda^\beta} \right), \quad a.e. y \in \mathbb{R}^n,$$

for any $\lambda \geq \lambda_0$. Here C depends only on Ω , T , M_1 and M_2 .

Proof. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a positive function which is supported in the unit ball $B(0, 1)$ and such that $\|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Define

$$(3.27) \quad \varphi_h(x) = h^{-n/2} \psi \left(\frac{x - y}{h} \right),$$

where $y \in \mathcal{A}_r$. Then, for $h > 0$ sufficiently small we can verify that

$$\text{Supp } \varphi_h \cap \Omega = \emptyset, \quad \text{and} \quad \text{Supp } \varphi_h \pm T\omega \cap \Omega = \emptyset.$$

Moreover, we have

$$\begin{aligned}
 & \left| \exp \left[-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right] - 1 \right| = \left| \int_{\mathbb{R}^n} \varphi_h^2(x) \left[\exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right] dx \right| \\
 & \leq \left| \int_{\mathbb{R}^n} \varphi_h^2(x) \left[\exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - \exp \left(-\frac{1}{2} \int_0^T a(x - s\omega, s) ds \right) \right] dx \right| \\
 (3.28) \quad & + \left| \int_{\mathbb{R}^n} \varphi_h^2(x) \left[\exp \left(-\frac{1}{2} \int_0^T a(x - s\omega, s) ds \right) - 1 \right] dx \right|.
 \end{aligned}$$

Therefore, since we have

$$\left| \exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - \exp \left(-\frac{1}{2} \int_0^T a(x - s\omega, s) ds \right) \right| \leq C \left| \int_0^T a(y - s\omega, s) - a(x - s\omega, s) ds \right|,$$

and using the fact that

$$\left| \int_0^T (a(y - s\omega, s) - a(x - s\omega, s)) ds \right| \leq C |y - x|,$$

we deduce upon applying Lemma 2.3 with $\varphi = \varphi_h$ the following estimation

$$\left| \exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right| \leq C \int_{\mathbb{R}^n} \varphi_h^2(x) |y - x| dx + C \left(\lambda^2 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda} \right) \|\varphi_h\|_{H^3(\mathbb{R}^n)}^2.$$

On the other hand, we have

$$\|\varphi_h\|_{H^3(\mathbb{R}^n)} \leq Ch^{-3} \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi_h^2(x) |y - x| dx \leq Ch.$$

So that we end up getting the following inequality

$$\left| \exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right| \leq Ch + C \left(\lambda^2 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda} \right) h^{-6}.$$

Selecting h small such that $h = 1/\lambda h^6$, that is $h = \lambda^{-1/7}$, we find two constants $\delta > 0$ and $\beta > 0$ such that

$$\left| \exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right| \leq C \left[\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda^\beta} \right].$$

Now, using the fact that $|X| \leq e^M |e^X - 1|$ for any $|X| \leq M$, we deduce that

$$\left| -\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right| \leq e^{M_1 T} \left| \exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right|.$$

Hence, we conclude that for all $y \in \mathcal{A}_r$ and $\omega \in \mathbb{S}^{n-1}$ we have

$$\left| \int_0^T a(y - s\omega, s) ds \right| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda^\beta} \right).$$

Since $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, this entails that for all $y \in \mathcal{A}_r$, and $\omega \in \mathbb{S}^{n-1}$, we have

$$(3.29) \quad \left| \int_{\mathbb{R}} a(y - t\omega, t) dt \right| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda^\beta} \right).$$

Moreover, if $y \in B(0, r/2)$, we have $|y - t\omega| \geq |t| - |y| \geq |t| - \frac{r}{2}$. Hence, one can see that $(y - t\omega, t) \notin \mathcal{C}_r^+$ if $t > r/2$. On the other hand, we have $(y - t\omega, t) \notin \mathcal{C}_r^+$ if $t \leq \frac{r}{2}$. Thus, we conclude that $(y - t\omega, t) \notin \mathcal{C}_r^+ \supset Q_{r,*}$ for $t \in \mathbb{R}$. This and the fact that $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, entails that for all $y \in B(0, r/2)$ and $\omega \in \mathbb{S}^{n-1}$, we have

$$a(y - t\omega, t) = 0, \quad \forall t \in \mathbb{R}.$$

By a similar way, we prove for $|y| \geq T - r/2$, that $(y - t\omega, t) \notin \mathcal{C}_r^- \supset Q_{r,*}$ for $t \in \mathbb{R}$ and then $a(y - t\omega, t) = 0$. Hence, we conclude that

$$(3.30) \quad \int_{\mathbb{R}} a(y - t\omega, t) dt = 0, \quad \text{a.e. } y \notin \mathcal{A}_r, \quad \omega \in \mathbb{S}^{n-1}.$$

Thus, by (3.29) and (3.30) we finish the proof of the lemma by getting

$$|\mathcal{R}(a)(y, \omega)| = \left| \int_{\mathbb{R}} a(t, y - t\omega) dt \right| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda^\beta} \right), \quad \text{a.e. } y \in \mathbb{R}^n, \quad \omega \in \mathbb{S}^{n-1}.$$

The proof of Lemma 3.2 is complete. \square

Our goal is to obtain an estimate linking the Fourier transform with respect to (x, t) of the absorbing coefficient $a = a_2 - a_1$ to the measurement $\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}$ in this set

$$(3.31) \quad E = \{(\xi, \tau) \in (\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}) \times \mathbb{R}, \quad |\tau| < |\xi|\}.$$

We denote by \widehat{F} the Fourier transform of a function $F \in L^1(\mathbb{R}^{n+1})$ with respect to (x, t) :

$$\widehat{F}(\xi, \tau) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} F(x, t) e^{-ix \cdot \xi} e^{-it\tau} dx dt.$$

We aim for proving that the Fourier transform of a is bounded as follows:

Lemma 3.3. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exist $C > 0$, $\delta > 0$, $\beta > 0$ and $\lambda_0 > 0$, such that the following estimate*

$$|\widehat{a}(\xi, \tau)| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda^\beta} \right),$$

holds for any $(\xi, \tau) \in E$ and $\lambda \geq \lambda_0$.

Proof. Let $(\xi, \tau) \in E$ and $\zeta \in \mathbb{S}^{n-1}$ be such that $\xi \cdot \zeta = 0$. Setting

$$\omega = \frac{\tau}{|\xi|^2} \cdot \xi + \sqrt{1 - \frac{\tau^2}{|\xi|^2}} \cdot \zeta.$$

Then, one can see that $\omega \in \mathbb{S}^{n-1}$ and $\omega \cdot \xi = \tau$. On the other hand by the change of variable $x = y - t\omega$, we have for all $\xi \in \mathbb{R}^n$ and $\omega \in \mathbb{S}^{n-1}$ the following identity

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{R}(a)(y, \omega) e^{-iy \cdot \xi} dy &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} a(y - t\omega, t) dt \right) e^{-iy \cdot \xi} dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} a(x, t) e^{-ix \cdot \xi} e^{-it\omega \cdot \xi} dx dt \\ &= \widehat{a}(\xi, \omega \cdot \xi) = \widehat{a}(\xi, \tau), \end{aligned}$$

where we have set $(\xi, \tau) = (\xi, \omega \cdot \xi) \in E$. Bearing in mind that for any $t \in \mathbb{R}$, $\text{Supp } a(\cdot, t) \subset \Omega \subset B(0, r/2)$, we deduce that

$$\int_{\mathbb{R}^n \cap B(0, \frac{r}{2} + T)} \mathcal{R}(a)(\omega, y) e^{-iy \cdot \xi} dy = \widehat{a}(\xi, \tau).$$

Then, in view of Lemma 3.2, we finish the proof of this lemma. \square

3.1.3. *End of the proof of Theorem 1.1.* We are now in position to complete the proof of Theorem 1.1, using the result we have already obtained and an analytic argument that is inspired by [1] and adapted for our case. For $\rho > 0$ and $\kappa \in (\mathbb{N} \cup \{0\})^{n+1}$, we put

$$|\kappa| = \kappa_1 + \dots + \kappa_{n+1}, \quad B(0, \rho) = \{x \in \mathbb{R}^{n+1}, |x| < \rho\}.$$

We state the following result which is proved in Appendix A (see also [31]).

Lemma 3.4. *Let \mathcal{O} be a non empty open set of the unit ball $B(0, 1) \subset \mathbb{R}^d$, $d \geq 2$, and let F be an analytic function in $B(0, 2)$, that satisfy*

$$\|\partial^\kappa F\|_{L^\infty(B(0,2))} \leq \frac{M|\kappa|!}{(2\rho)^{|\kappa|}}, \quad \forall \kappa \in (\mathbb{N} \cup \{0\})^d,$$

for some $M > 0$, $\rho > 0$ and $N = N(\rho)$. Then, we have

$$\|F\|_{L^\infty(B(0,1))} \leq NM^{1-\gamma} \|F\|_{L^\infty(\mathcal{O})}^\gamma,$$

where $\gamma \in (0, 1)$ depends on d , ρ and $|\mathcal{O}|$.

The above lemma claims conditional stability for the analytic continuation. For classical results for this type, one can see Lavrent'ev, Romanov and Shishatskiĭ [21]. For a fixed $\alpha > 0$, we set $F_\alpha(\tau, \xi) = \hat{a}(\alpha(\xi, \tau))$, for all $(\xi, \tau) \in \mathbb{R}^{n+1}$. It is easy to see that F_α is analytic and we have

$$\begin{aligned} |\partial^\kappa F_\alpha(\xi, \tau)| &= |\partial^\kappa \hat{a}(\alpha(\xi, \tau))| = \left| \partial^\kappa \int_{\mathbb{R}^{n+1}} a(x, t) e^{-i\alpha(t, x) \cdot (\xi, \tau)} dx dt \right| \\ &= \left| \int_{\mathbb{R}^{n+1}} a(x, t) (-i)^{|\kappa|} \alpha^{|\kappa|}(x, t)^\kappa e^{-i\alpha(x, t) \cdot (\xi, \tau)} dx dt \right|. \end{aligned}$$

This entails that

$$|\partial^\kappa F_\alpha(\xi, \tau)| \leq \int_{\mathbb{R}^{n+1}} |a(x, t)| \alpha^{|\kappa|} (|x|^2 + t^2)^{\frac{|\kappa|}{2}} dx dt \leq \|a\|_{L^1(Q_{r,*})} \alpha^{|\kappa|} (2T^2)^{\frac{|\kappa|}{2}} \leq C \frac{|\kappa|!}{(T^{-1})^{|\kappa|}} e^\alpha.$$

The, upon applying Lemma 3.4 with $M = Ce^\alpha$, $2\rho = T^{-1}$, and $\mathcal{O} = \mathring{E} \cap B(0, 1)$, where

$$\mathring{E} = \{(\xi, \tau) \in \mathbb{R} \times (\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}), |\tau| < |\xi|\},$$

one may find a constant $\mu \in (0, 1)$ such that we have for all $(\xi, \tau) \in B(0, 1)$, the following estimation

$$|F_\alpha(\xi, \tau)| = |\hat{a}(\alpha(\xi, \tau))| \leq Ce^{\alpha(1-\gamma)} \|F_\alpha\|_{L^\infty(\mathcal{O})}^\gamma.$$

Now the idea is to find an estimate for the Fourier transform of a in a suitable ball. Using the fact that $\alpha \mathring{E} = \{\alpha(\xi, \tau), (\xi, \tau) \in \mathring{E}\} = \mathring{E}$, we obtain for all $(\xi, \tau) \in B(0, \alpha)$

$$\begin{aligned} |\hat{a}(\xi, \tau)| &= |F_\alpha(\alpha^{-1}(\xi, \tau))| \leq Ce^{\alpha(1-\gamma)} \|F_\alpha\|_{L^\infty(\mathcal{O})}^\gamma \\ &\leq Ce^{\alpha(1-\gamma)} \|\hat{a}\|_{L^\infty(B(0, \alpha) \cap \mathring{E})}^\mu \\ (3.32) \quad &\leq Ce^{\alpha(1-\gamma)} \|\hat{a}\|_{L^\infty(\mathring{E})}^\gamma. \end{aligned}$$

The next step in the proof is to deduce an estimate that links the unknown coefficient a to the measurement $\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}$. To obtain such estimate, we need first to decompose the $H^{-1}(\mathbb{R}^{n+1})$ norm of a into the following way

$$\begin{aligned} \|a\|_{H^{-1}(\mathbb{R}^{n+1})}^{2/\gamma} &= \left(\int_{|(\tau, \xi)| < \alpha} (1 + |(\tau, \xi)|^2)^{-1} |\hat{a}(\xi, \tau)|^2 d\xi d\tau + \int_{|(\xi, \tau)| \geq \alpha} (1 + |(\tau, \xi)|^2)^{-1} |\hat{a}(\xi, \tau)|^2 d\xi d\tau \right)^{1/\gamma} \\ &\leq C \left(\alpha^{n+1} \|\hat{a}\|_{L^\infty(B(0, \alpha))}^2 + \alpha^{-2} \|a\|_{L^2(\mathbb{R}^{n+1})}^2 \right)^{1/\gamma}. \end{aligned}$$

Hence, in light of (3.32) and Lemma 3.3, we get

$$\begin{aligned} \|a\|_{H^{-1}(\mathbb{R}^{n+1})}^{2/\gamma} &\leq C \left(\alpha^{n+1} e^{2\alpha(1-\gamma)} (\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \frac{1}{\lambda^\beta})^{2\gamma} + \alpha^{-2} \right)^{1/\gamma} \\ &\leq C \left(\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \lambda^{2\beta} \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\|^2 + \alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \lambda^{-2\beta} + \alpha^{-2/\gamma} \right). \end{aligned}$$

Let $\alpha_0 > 0$ be sufficiently large and assume that $\alpha > \alpha_0$. Setting

$$\lambda = \alpha^{\frac{n+3}{2\mu\gamma}} e^{\frac{\alpha(1-\mu)}{\mu\gamma}},$$

and using the fact that $\alpha > \alpha_0$, one can see that $\lambda > \lambda_0$ and $\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \lambda^{-2\beta} = \alpha^{-2/\gamma}$. This entails that

$$\begin{aligned} \|a\|_{H^{-1}(\mathbb{R}^{n+1})}^{2/\gamma} &\leq C \left(\alpha^{\frac{\beta(n+1)+\delta(n+3)}{\beta\gamma}} e^{\frac{2\alpha(\beta+\delta)(1-\gamma)}{\beta\gamma}} \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\|^2 + \alpha^{-2/\gamma} \right) \\ &\leq C \left(e^{N\alpha} \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\|^2 + \alpha^{-2/\gamma} \right), \end{aligned}$$

where N depends on δ , β , n , and γ . The next step is to minimize the right hand-side of the above inequality with respect to α . We need to take α sufficiently large. So, there exists a constant $m > 0$ such that if $0 < \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| < m$, and

$$\alpha = \frac{1}{N} |\log \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\||,$$

then, we have the following estimation

$$\begin{aligned} \|a\|_{H^{-1}(Q_{r,*})} &\leq \|a\|_{H^{-1}(\mathbb{R}^{n+1})} \leq C \left(\|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + |\log \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\||^{-2/\gamma} \right)^{\gamma/2} \\ (3.33) \quad &\leq C \left(\|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\|^{\gamma/2} + |\log \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\||^{-1} \right). \end{aligned}$$

Now if $\|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| \geq m$, we have

$$\|a\|_{H^{-1}(Q_{r,*})} \leq C \|a\|_{L^\infty(Q_{r,*})} \leq \frac{2CMc^{\gamma/2}}{m^{\gamma/2}} \leq \frac{2CM}{m^{\gamma/2}} \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\|^{\gamma/2},$$

hence (3.33) holds. Let us now consider $\theta > 1$ such that $p := s - 1 = \frac{n+1}{2} + 2\theta$. Use Sobolev's embedding theorem we find by interpolating

$$\begin{aligned} \|a\|_{L^\infty(Q_{r,*})} &\leq C \|a\|_{H^{\frac{n}{2}+\theta}(Q_{r,*})} \\ &\leq C \|a\|_{H^{-1}(Q_{r,*})}^{1-\eta} \|a\|_{H^{s-1}(Q_{r,*})}^\eta \\ &\leq C \|a\|_{H^{-1}(Q_{r,*})}^{1-\eta}, \end{aligned}$$

for some $\eta \in (0, 1)$. This completes the proof of Theorem 1.1. This will be a key ingredient in proving the result of the next section.

3.2. Stability for the potential. This section is devoted to the proof of Theorem 1.2. By means of the geometrical optics solutions constructed in Section 2, we will show using the stability estimate we have already obtained for the absorbing coefficient a , that the time dependent potential b depends stably on the Dirichlet-to-Neumann map $\Lambda_{a,b}$. As before, given $\omega \in \mathbb{S}^{n-1}$, $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$ such that $(a_1, b_1) = (a_2, b_2)$ in $\overline{Q_r} \setminus Q_{r,*}$, we set

$$a = a_2 - a_1, \quad b = b_2 - b_1 \quad \text{and} \quad A(x, t) = (A^- A^+)(x, t) = \exp \left(-\frac{1}{2} \int_0^t a(x + (t-s)\omega, s) ds \right),$$

where A^- and A^+ are given by

$$A^-(x, t) = \exp \left(\frac{1}{2} \int_0^t a_1(x + (t-s)\omega, s) ds \right), \quad A^+(x, t) = \exp \left(-\frac{1}{2} \int_0^t a_2(x + (t-s)\omega, s) ds \right).$$

In the rest of this section, we define b in \mathbb{R}^{n+1} by $b = b_2 - b_1$ in $\overline{Q_r}$ and $b = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q_r}$. We start by giving a preliminary estimate that will be used to prove the main statement of this section.

Lemma 3.5. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exists $C > 0$ such that for any $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in \mathcal{C}_0^\infty(\mathcal{A}_r)$, the following estimate holds*

$$\left| \int_0^T \int_{\mathbb{R}^n} b(y - t\omega, t) \varphi^2(y) dy dt \right| \leq C \left(\lambda^3 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2,$$

for any $\lambda > 0$ sufficiently large. Here C depends only on Ω , M_1 , M_2 and T .

Proof. We start with the identity (3.23), except this time we will isolate the electric potential

$$\int_0^T \int_{\Omega} b(x, t) u^+ u^- dx dt = \int_0^T \int_{\Gamma} (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt - \int_0^T \int_{\Omega} a(x, t) \partial_t u^+ u^- dx dt.$$

By replacing u^+ and u^- by their expressions we get

$$\begin{aligned} \int_0^T \int_{\Omega} b(x, t) \varphi^2(x + t\omega) A(x, t) dx dt &= \int_0^T \int_{\Gamma} (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt \\ &- \int_0^T \int_{\Omega} b(x, t) \varphi(x + t\omega) A^-(x, t) e^{-i\lambda(x \cdot \omega + t)} r_\lambda^+(x, t) dx dt - \int_0^T \int_{\Omega} b(x, t) r_\lambda^+(x, t) r_\lambda^-(x, t) dx dt \\ &- \int_0^T \int_{\Omega} a(x, t) \partial_t u^+ u^- dx dt - \int_0^T \int_{\Omega} b(x, t) \varphi(x + t\omega) A^+(x, t) e^{i\lambda(x \cdot \omega + t)} r^-(x, t) dx dt \\ (3.34) \qquad \qquad \qquad &= \int_0^T \int_{\Gamma} (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt + I'_\lambda. \end{aligned}$$

Then, in view of (3.34), we have

$$\int_0^T \int_{\Omega} b(x, t) \varphi^2(x + t\omega) dx dt = \int_0^T \int_{\Omega} b(x, t) \varphi^2(x + t\omega) (1 - A) dx dt + \int_0^T \int_{\Gamma} (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt + I'_\lambda.$$

From (2.11), (2.19) and using the fact that $a = a_2 - a_1 = 0$ outside $Q_{r,*}$, we find

$$(3.35) \qquad |I'_\lambda| \leq C \left(\lambda \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

By the trace theorem, we get

$$\begin{aligned} \left| \int_0^T \int_{\Gamma} (\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1})(f_\lambda) u^- d\sigma dt \right| &\leq \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| \|f_\lambda\|_{H^1(\Sigma)} \|u^-\|_{L^2(\Sigma)} \\ (3.36) \qquad \qquad \qquad &\leq C \lambda^3 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| \|\varphi\|_{H^3(\mathbb{R}^n)}^2. \end{aligned}$$

On the other hand, we have

$$(3.37) \qquad \left| \int_0^T \int_{\Omega} b(x, t) \varphi^2(x + t\omega) (1 - A) dx dt \right| \leq C \|a\|_{L^\infty(Q_{r,*})} \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

Then, in light of (3.35)-(3.37), taking to account that $b = b_2 - b_1 = 0$ outside $Q_{r,*}$ and using the change of variables $y = x + t\omega$ we get

$$\left| \int_0^T \int_{\mathbb{R}^n} b(y - t\omega, t) \varphi^2(y) dy dt \right| \leq C \left(\lambda^3 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

This completes the proof of the Lemma. \square

Now the idea is to deduce an estimate for the light ray transform of the time-dependent unknown coefficient b in order to control thereafter its Fourier transform.

Lemma 3.6. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exists $C > 0$, $\delta > 0$, $\beta > 0$ and $\lambda_0 > 0$ such that for all $\omega \in \mathbb{S}^{n-1}$, the following estimate holds*

$$\left| \mathcal{R}(b)(y, \omega) \right| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda^\delta \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta} \right), \quad a. e. y \in \mathbb{R}^n,$$

for any $\lambda > \lambda_0$. Here C depends only on Ω , T , M_1 and M_2 .

Proof. We proceed as in the proof of Lemma 3.2. We consider the sequence $(\varphi_h)_h$ defined by (3.27) with $y \in \mathcal{A}_r$. Since we have

$$\begin{aligned} \left| \int_0^T b(y - t\omega, t) dt \right| &= \left| \int_0^T \int_{\mathbb{R}^n} b(y - t\omega, t) \varphi_h^2(x) dx dt \right| \\ &\leq \left| \int_0^T \int_{\mathbb{R}^n} b(x - t\omega, t) \varphi_h^2(x) dx dt \right| + \left| \int_0^T \int_{\mathbb{R}^n} (b(y - t\omega, t) - b(x - t\omega, t)) \varphi_h^2(x) dx dt \right|. \end{aligned}$$

Then, by applying Lemma 3.5 with $\varphi = \varphi_h$, and since $|b(y - t\omega, t) - b(x - t\omega, t)| \leq C|y - x|$, we obtain

$$\left| \int_0^T b(y - t\omega, t) dt \right| \leq C \left(\lambda^3 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda} \right) \|\varphi_h\|_{H^3(\mathbb{R}^n)}^2 + C \int_{\mathbb{R}^n} |x - y| \varphi_h^2(x) dx.$$

On the other hand, since $\|\varphi_h\|_{H^3(\mathbb{R}^n)} \leq Ch^{-3}$ and $\int_{\mathbb{R}^n} |x - y| \varphi_h^2(x) dx \leq Ch$, we conclude that

$$\left| \int_0^T b(y - t\omega, t) dt \right| \leq C \left(\lambda^3 \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda} \right) h^{-6} + Ch.$$

Selecting h small such that $h = h^{-6}/\lambda$. Then, we find two constants $\delta > 0$ and $\beta > 0$ such that

$$\left| \int_0^T b(y - t\omega, t) dt \right| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda^\delta \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta} \right).$$

Using the fact that $b = b_2 - b_1 = 0$ outside $Q_{r,*}$, we then conclude that for all $y \in \mathcal{A}_r$ and $\omega \in \mathbb{S}^{n-1}$,

$$\left| \int_{\mathbb{R}} b(y - t\omega, t) dt \right| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda^\delta \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta} \right).$$

Next, by arguing as in the derivation of Lemma 3.2, we end up upper bounding the light-ray transform of b , for all $y \in \mathbb{R}^n$. \square

At this point, it is convenient to recall that our goal is to obtain an estimate for the Fourier transform of b in a precise set. So, by proceeding by a similar way as in the previous section, we get this result.

Lemma 3.7. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exists $C > 0$, $\delta > 0$, $\beta > 0$ and $\lambda_0 > 0$, such that the following estimate*

$$|\hat{b}(\xi, \tau)| \leq C \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda^\delta \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta} \right), \quad a. e. (\xi, \tau) \in E,$$

for any $\lambda > \lambda_0$. Here C depends only on Ω , T , M_1 and M_2 .

Next, using the above estimation as well as the analytic continuation argument, that is Lemma 3.4, we upper bound the Fourier transform of b in a suitable ball $B(0, \alpha)$ as follows

$$(3.38) \quad |\hat{b}(\xi, \tau)| \leq C e^{\alpha(1-\gamma)} \left(\lambda^\delta \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| + \lambda^\delta \|a\|_{L^\infty(Q_{r,*})} + \frac{1}{\lambda^\beta} \right)^\gamma,$$

for some $\gamma \in (0, 1)$ and where $\alpha > 0$ is assumed to be sufficiently large. Then, in order to deduce an estimate linking the unknown coefficient b to the measurement $\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}$, we control the $H^{-1}(\mathbb{R}^{n+1})$ norm of b as follows

$$\|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C \left[\alpha^{n+1} \|\widehat{b}\|_{L^\infty(B(0, \alpha))}^2 + \alpha^{-2} \|b\|_{L^2(\mathbb{R}^{n+1})}^2 \right]^{\frac{1}{\gamma}}.$$

So, by the use of (3.38), we obtain the following inequality

$$(3.39) \quad \|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C \left[\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \left(\lambda^{2\delta} \epsilon^2 + \lambda^{2\delta} \|a\|_{L^\infty(Q_{r,*})}^2 + \lambda^{-2\beta} \right) + \alpha^{-\frac{2}{\gamma}} \right],$$

where we have set $\epsilon = \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\|$. In light of Theorem 1.1, one gets

$$\|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C \left[\alpha^{\frac{n+1}{\gamma}} e^{\frac{2\alpha(1-\gamma)}{\gamma}} \left(\lambda^{2\delta} \epsilon^2 + \lambda^{2\delta} \epsilon^{2\mu_1 \mu_2} + \lambda^{2\delta} |\log \epsilon|^{-2\mu_2} + \lambda^{-2\beta} \right) + \alpha^{-\frac{2}{\gamma}} \right],$$

for some $\gamma, \mu_1, \mu_2 \in (0, 1)$ and $\delta, \beta > 0$. Let $\alpha_0 > 0$ be sufficiently large and we take $\alpha > \alpha_0$. Setting

$$\lambda = \alpha^{\frac{n+3}{2\gamma\beta}} e^{\frac{\alpha(1-\gamma)}{\gamma\beta}}.$$

By $\alpha > \alpha_0$, we can assume $\lambda > \lambda_0$. Therefore, the estimate (3.39) yields

$$\|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C \left[e^{N\alpha} (\epsilon^2 + \epsilon^{2s} + |\log \epsilon|^{-2\mu_2}) + \alpha^{-\frac{2}{\gamma}} \right],$$

for some $s, \mu_1, \mu_2 \in (0, 1)$, and where N is depending on n, γ, δ and β . Thus, if ϵ is small, we have

$$(3.40) \quad \|b\|_{H^{-1}(\mathbb{R}^{n+1})}^{\frac{2}{\gamma}} \leq C \left(e^{N\alpha} |\log \epsilon|^{-2\mu_2} + \alpha^{-\frac{2}{\gamma}} \right).$$

In order to minimize the right hand side of the above inequality with respect to α , we need to take α sufficiently large. So, we select α as follows

$$\alpha = \frac{1}{N} \log |\log \epsilon|^{\mu_2},$$

where we have assumed that $\epsilon < c \leq 1$. Then, the estimate (3.40) yields

$$\|b\|_{H^{-1}(Q_{r,*})} \leq \|b\|_{H^{-1}(\mathbb{R}^{n+1})} \leq C \left(\log |\log \|\Lambda_{a_2, b_2} - \Lambda_{a_1, b_1}\| |^{\mu_2} \right)^{-1}.$$

This completes the proof of Theorem 1.2.

4. DETERMINATION OF COEFFICIENTS FROM BOUNDARY MEASUREMENTS AND FINAL DATA

In this section, we prove Theorem 1.4 and 1.5. We will extend the stability estimates obtained in the first case to a larger region $Q_{r,\#} \supset Q_{r,*}$. We shall consider the geometric optics solutions constructed in Section 2, associated with a function φ obeying $\text{supp } \varphi \cap \Omega = \emptyset$. Note that this time, we have more flexibility on the support of the function φ and we don't need to assume that $\text{supp } \varphi \pm T\omega \cap \Omega = \emptyset$ anymore. We recall that the observations in this case are given by the following operator

$$\begin{aligned} \mathcal{R}_{a,b} : \mathcal{H}_0^1(\Sigma) &\longrightarrow \mathcal{K} \\ f &\longmapsto (\partial_\nu u, u(\cdot, T), \partial_t u(\cdot, T)), \end{aligned}$$

associated to the problem (1.1) with $(u_0, u_1) = (0, 0)$. We denote by

$$\mathcal{R}_{a,b}^1(f) = \partial_\nu u, \quad \mathcal{R}_{a,b}^2(f) = u(\cdot, T), \quad \mathcal{R}_{a,b}^3(f) = \partial_t u(\cdot, T).$$

4.1. Stability for the absorbing coefficient. In this section we will prove that the absorbing coefficient a can be stably recovered in a larger region if we further know the final data of the solution u of the dissipative wave equation (1.1). In the rest of this section, we define $a = a_2 - a_1$ in \overline{Q}_r and $a = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q}_r$. We shall first prove the following statement

Lemma 4.1. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be such that $\text{supp } \varphi \cap \Omega = \emptyset$. There exists $C > 0$, such that for any $\omega \in \mathbb{S}^{n-1}$, the following estimate holds*

$$\left| \int_{\mathbb{R}^n} \varphi^2(y) \left[\exp \left(-\frac{1}{2} \int_0^T a(y - s\omega, s) ds \right) - 1 \right] dy \right| \leq C \left(\lambda^2 \|\mathcal{R}_{a_2, b_2} - \mathcal{R}_{a_1, b_1}\| + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

Here C depends only on Ω , T , M_1 and M_2 .

Proof. In view of Lemma 2.1 and using the fact that $\text{supp } \varphi \cap \Omega = \emptyset$, there exists a geometrical optic solution u^+ to the wave equation

$$\begin{cases} \left(\partial_t^2 - \Delta + a_2(x, t) \partial_t + b_2(x, t) \right) u^+ = 0 & \text{in } Q, \\ u^+(x, 0) = \partial_t u^+(x, 0) = 0 & \text{in } \Omega, \end{cases}$$

in the following form

$$(4.41) \quad u^+(x, t) = \varphi(x + t\omega) A^+(x, t) e^{i\lambda(x \cdot \omega + t)} + r_\lambda^+(x, t),$$

corresponding to the coefficients a_2 and b_2 , where $r_\lambda^+(x, t)$ satisfies (2.10) and (2.11). We denote

$$f_\lambda(x, t) = u^+(x, t)|_\Sigma = \varphi(x + t\omega) A^+(x, t) e^{i\lambda(x \cdot \omega + t)}.$$

Let u_1 be the solution of

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + a_1(x, t) \partial_t u_1 + b_1(x, t) u_1 = 0 & \text{in } Q, \\ u_1(x, 0) = \partial_t u_1(x, 0) = 0 & \text{in } \Omega, \\ u_1 = f_\lambda & \text{on } \Sigma. \end{cases}$$

Putting $u = u_1 - u^+$. Then, u is a solution to the following system

$$(4.42) \quad \begin{cases} \partial_t^2 u - \Delta u + a_1(x, t) \partial_t u + b_1(x, t) u = a(x, t) \partial_t u^+ + b(x, t) u^+ & \text{in } Q, \\ u(x, 0) = \partial_t u(x, 0) = 0 & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \Sigma, \end{cases}$$

where $a = a_2 - a_1$ and $b = b_2 - b_1$. On the other hand, Lemma 2.3 guarantees the existence of a geometrical optic solution u^- to the adjoint problem

$$\partial_t^2 u^- - \Delta u^- - a_1(x, t) \partial_t u^- + (b_1(x, t) - \partial_t a_1(x, t)) u^- = 0 \quad \text{in } Q,$$

corresponding to the coefficients a_1 and $(-\partial_t a_1 + b_1)$, in the form

$$(4.43) \quad u^-(x, t) = \varphi(x + t\omega) e^{-i\lambda(x \cdot \omega + t)} A^-(x, t) + r_\lambda^-(x, t),$$

where $r_\lambda^-(x, t)$ satisfies (2.18) and (2.19). Multiplying the first equation of (4.42) by u^- , integrating by parts and using Green's formula, we get

$$\int_0^T \int_\Omega a(x, t) \partial_t u^+ u^- dx dt = \int_0^T \int_\Gamma (\mathcal{R}_{a_2, b_2}^1 - \mathcal{R}_{a_1, b_1}^1)(f_\lambda) u^-(x, t) d\sigma dt - \int_\Omega (\mathcal{R}_{a_2, b_2}^3 - \mathcal{R}_{a_1, b_1}^3)(f_\lambda) u^-(x, T) dx$$

$$(4.44) \quad \begin{aligned} & - \int_{\Omega} (\mathcal{R}_{a_2, b_2}^2 - \mathcal{R}_{a_1, b_1}^2)(f_{\lambda}) \left[a_1(x, T) u^-(x, T) - \partial_t u^-(x, T) \right] dx \\ & - \int_0^T \int_{\Omega} b(x, t) u^+(x, t) u^-(x, t) dx dt. \end{aligned}$$

By replacing u^+ and u^- by their expressions, using (3.25) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \left| \int_0^T \int_{\Omega} a(x, t) \varphi^2(x + t\omega) A(x, t) dx dt \right| & \leq \frac{C}{\lambda} \left[\left(\|u^-\|_{L^2(\Sigma)}^2 + \|u^-(\cdot, T)\|_{L^2(\Omega)}^2 + \|\partial_t u^-(\cdot, T)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \right. \\ & \left. \left(\|(\mathcal{R}_{a_2, b_2}^1 - \mathcal{R}_{a_1, b_1}^1)(f_{\lambda})\|_{L^2(\Sigma)}^2 + \|(\mathcal{R}_{a_2, b_2}^2 - \mathcal{R}_{a_1, b_1}^2)(f_{\lambda})\|_{H^1(\Omega)}^2 + \|(\mathcal{R}_{a_2, b_2}^3 - \mathcal{R}_{a_1, b_1}^3)(f_{\lambda})\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} + \|\varphi\|_{H^3(\mathbb{R}^n)}^2 \right]. \end{aligned}$$

Then, by setting $\phi_{\lambda} = (u_{|\Sigma}^-, u^-(\cdot, T), \partial_t u^-(\cdot, T))$, one can see that

$$\left| \int_0^T \int_{\Omega} a(x, t) \varphi^2(x + t\omega) A(x, t) dx dt \right| \leq \frac{C}{\lambda} \left(\|\mathcal{R}_{a_2, b_2} - \mathcal{R}_{a_1, b_1}\| \|f_{\lambda}\|_{H^1(\Sigma)} \|\phi_{\lambda}\|_{\mathcal{K}} + \|\varphi\|_{H^3(\mathbb{R}^n)}^2 \right).$$

Therefore, by the trace theorem we get

$$\left| \int_0^T \int_{\Omega} a(x, t) \varphi^2(x + t\omega) A(x, t) dx dt \right| \leq C \left(\lambda^2 \|\mathcal{R}_{a_2, b_2} - \mathcal{R}_{a_1, b_1}\| + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

Finally, we use the fact that $a = a_2 - a_1 = 0$ outside $Q_{r, \#}$ and we complete the proof of the lemma by arguing as in the proof of Lemma 3.1. \square

Next, by considering the sequence φ_h defined by (3.27) with $y \notin \Omega$, taking to account that $a = a_2 - a_1 = 0$ outside $Q_{r, \#}$ and arguing as in Section 3.1, we complete the proof of Theorem 1.4.

4.2. Stability for the potential. We are now in position to prove Theorem 1.5. We aim to show by the use of Theorem 1.4, that the potential b can be stably recovered in the region $Q_{r, \#}$, with respect to the operator $\mathcal{R}_{a, b}$. In the rest of this section, we define b in \mathbb{R}^{n+1} by $b = b_2 - b_1$ in \overline{Q}_r and $b = 0$ on $\mathbb{R}^{n+1} \setminus \overline{Q}_r$.

Lemma 4.2. *Let $(a_i, b_i) \in \mathcal{A}(M_1, M_2)$, $i = 1, 2$. There exists $C > 0$ such that for any $\omega \in \mathbb{S}^{n-1}$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp } \varphi \cap \Omega = \emptyset$, the following estimate holds*

$$\left| \int_0^T \int_{\mathbb{R}^n} b(y - t\omega, t) \varphi^2(y) dy dt \right| \leq C \left(\lambda^3 \|\mathcal{R}_{a_2, b_2} - \mathcal{R}_{a_1, b_1}\| + \lambda \|a\|_{L^\infty(Q_{r, \#})} + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2,$$

where C depends only on Ω , M_1 , M_2 and T .

Proof. We start with the identity (4.44), except this time we isolate the potential b , we get

$$\begin{aligned} \int_0^T \int_{\Omega} b(x, t) u^+ u^- dx dt & = \int_0^T \int_{\Gamma} (\mathcal{R}_{a_2, b_2}^1 - \mathcal{R}_{a_1, b_1}^1)(f_{\lambda}) u^-(x, t) d\sigma dt - \int_{\Omega} (\mathcal{R}_{a_2, b_2}^3 - \mathcal{R}_{a_1, b_1}^3)(f_{\lambda}) u^-(x, T) dx \\ & \quad - \int_{\Omega} (\mathcal{R}_{a_2, b_2}^2 - \mathcal{R}_{a_1, b_1}^2)(f_{\lambda}) \left[a_1(x, T) u^-(x, T) - \partial_t u^-(x, T) \right] dx \\ & \quad - \int_0^T \int_{\Omega} a(x, t) \partial_t u^+(x, t) u^-(x, t) dx dt. \end{aligned}$$

So, by replacing u^+ and u^- by their expressions, taking to account (3.35), (3.37) and the fact that $a = a_2 - a_1 = 0$ outside $Q_{r, \#}$, and making the change of variables $y = x + t\omega$, we obtain

$$\left| \int_0^T \int_{\mathbb{R}^n} b(y - t\omega, t) \varphi^2(y) dy dt \right| \leq C \left(\lambda^3 \|\mathcal{R}_{a_2, b_2} - \mathcal{R}_{a_1, b_1}\| + \lambda \|a\|_{L^\infty(Q_{r, \#})} + \frac{1}{\lambda} \right) \|\varphi\|_{H^3(\mathbb{R}^n)}^2.$$

This completes the proof of the lemma. \square

In order to complete the proof of Theorem 1.5, it will be enough to consider the sequence (φ_h) defined by (3.27), with, $y \notin \Omega$, use the fact $b = b_2 - b_1 = 0$ outside $Q_{r,\sharp}$ and repeat the same arguments of Section 3.2

5. DETERMINATION OF COEFFICIENTS FROM BOUNDARY MEASUREMENTS AND FINAL DATA BY VARYING THE INITIAL DATA

In the present section, we deal with the same inverse problem, except the set of data, in this case, is made of the responses of the medium for all possible initial data. For $(a_i, b_i) \in \mathcal{C}^2(\overline{Q}) \times \mathcal{C}^1(\overline{Q})$, $i = 1, 2$, we define $(a, b) = (a_2 - a_1, b_2 - b_1)$ in Q and $(a, b) = (0, 0)$ on $\mathbb{R}^{n+1} \setminus Q$. By proceeding as in the derivation of Theorem 1.1 and Theorem 1.4, we prove a log-type stability estimate in the determination of the absorbing coefficient a over the whole domain Q , from the knowledge of the measurement $\mathcal{I}_{a,b}$.

To prove such estimate, we proceed as in Section 3.1 and 4.1, except this time, we have more flexibility on the support of the function φ_h defined by (3.27). Namely, we don't need to impose any condition on its support anymore (we fix $y \in \mathbb{R}^n$).

The same thing for the determination of the time-dependent potential b . we argue as in Section 3.2 and 4.2 to prove a log-log-type stability estimate in recovering the time dependent coefficient b with respect to the operator $\mathcal{I}_{a,b}$, over the whole domain Q .

APPENDIX A. PROOF OF LEMMA 3.4

In this section, we give the proof of Lemma 3.4 for analytic continuation. The proof is inspired from estimates given in [1][Theorem 3] for one variable analytic function. We simplify an adapted proof for our case. In order to express the main goal of this section we first prove the following Lemma

Lemma A.1. *Let J be an open interval in $[-\frac{1}{5}, \frac{1}{5}]$, and g be an holomorphic function in the unit disc $D(0, 1) \subset \mathbb{C}$ satisfying*

$$(A.45) \quad |g(z)| \leq 1, \quad |z| < 1.$$

Then, there exist $\gamma \in (0, 1)$ and $N > 0$ such that the following estimate holds

$$\|g\|_{L^\infty(B(0, \frac{1}{2}))} \leq N \|g\|_{L^\infty(J)}^\gamma,$$

where N and γ are depending only on $|J|$.

Proof. We should first notice that for all $n \geq 1$, there exist $(n + 1)$ points such that

$$-\frac{1}{5} \leq x_0 < \dots < x_n \leq \frac{1}{5},$$

with $x_i \in \overline{J}$, $i = 0, \dots, n$, and satisfying the following estimation

$$(A.46) \quad x_i - x_{i-1} \geq \frac{|J|}{n+1}, \quad \text{for } i = 1, \dots, n.$$

Let $z \in \mathbb{C}$. We denote by

$$P_n(z) = \sum_{i=0}^n g(x_i) \prod_{j \neq i} (z - x_j) \prod_{j \neq i} (x_i - x_j)^{-1}.$$

In order to prove this lemma, we need first to find an upper bound for $|P_n(z)|$. To do that we first notice that

for $l' > l$ we have $x_{l'} - x_l = \sum_{i=l+1}^{l'} (x_i - x_{i-1})$. Hence, (A.46) entails that

$$\begin{cases} (x_j - x_i) \geq (j - i) \frac{|J|}{n+1} & j > i, \\ (x_i - x_j) \geq (i - j) \frac{|J|}{n+1} & j < i. \end{cases}$$

As a consequence we have the following estimation

$$(A.47) \quad \prod_{j \neq i} |x_i - x_j| \geq \prod_{j=0}^{i-1} (i - j) \frac{|J|}{(n+1)} \prod_{j=i+1}^n (j - i) \frac{|J|}{(n+1)} \geq i! \frac{|J|^i}{(n+1)^i} (n - i)! \frac{|J|^{n-i}}{(n+1)^{n-i}}.$$

On the other hand, it is easy to see that for $|z| \leq \frac{1}{2}$ and $x_j \in \overline{J}$, $j = 0, \dots, n$, we have

$$\prod_{j \neq i} |z - x_j| \leq \prod_{j \neq i} (|z| + |x_j|) \leq 1,$$

Putting this together with (A.47), we end up getting this result

$$(A.48) \quad |P_n(z)| \leq \sum_{i=0}^n C_n^i \frac{(n+1)^n}{n! |E|^n} \|g\|_{L^\infty(J)} \leq e \left(\frac{6}{|J|} \right)^n \|g\|_{L^\infty(J)}.$$

The next step of the proof is to control $|g(z) - P_n(z)|$. For this purpose, let us introduce the following function: for all $\xi \in \mathbb{C}$, such that $|\xi| = 1$, we denote by

$$G(\xi) = g(\xi)(\xi - z)^{-1} \prod_{j=0}^n (\xi - x_j)^{-1}.$$

Applying the residue Theorem, one obtains the following identity

$$\frac{1}{2i\pi} \int_{|\xi|=1} G(\xi) d\xi = \left(\text{Res}(G, z) + \sum_{k=0}^n \text{Res}(G, x_k) \right) = (g(z) - P_n(z)) \prod_{j=0}^n (z - x_j)^{-1}.$$

From this and the hypothesis (A.45), it follows that for $|z| \leq \frac{1}{2}$, and $x_i \in \overline{J}$, we have

$$(A.49) \quad |g(z) - P_n(z)| \leq 2 \left(\frac{1}{2} + \frac{1}{5} \right)^{n+1} \left(1 - \frac{1}{5} \right)^{-(n+1)} \leq 2 \left(\frac{7}{8} \right)^n.$$

Combining (A.48) with (A.49), one gets

$$\|g\|_{L^\infty(B(0,1/2))} \leq 2 \left(\frac{7}{8} \right)^n + e \left(\frac{6}{|J|} \right)^n \|g\|_{L^\infty(J)}, \quad n \geq 1.$$

To complete the proof of the lemma, we need to minimize the right hand side of the last estimate with respect to n . To this end, let us define the following function

$$\psi(x) = 2e^{-x \log(8/7)} + e \|g\|_{L^\infty(J)} e^{x \log(6/|J|)}, \quad x \in \mathbb{R}.$$

A simple calculation show that the function ψ reaches a minimum at this point

$$x_0 = \left[\log \left(\frac{48}{7|J|} \right) \right]^{-1} \log \left[\frac{\log(8/7)}{e \|g\|_{L^\infty(J)} \log(6/|J|)} \right].$$

Then, we end up getting the desired result. □

We move now to establish the second result by the use of Hadamard's three-circle theorem and Lemma A.1.

Lemma A.2. *Let φ be an analytic function in $[-1, 1]$, and I an open interval in $[-1, 1]$. We assume that there exist positive constants M and ρ such that*

$$(A.50) \quad |\varphi^{(k)}(s)| \leq \frac{Mk!}{(2\rho)^k}, \quad k \geq 0, \quad s \in [-1, 1].$$

Then, there exist $N = N(\rho, |I|)$ and $\gamma = \gamma(\rho, |I|)$ such that we have

$$(A.51) \quad |\varphi(s)| \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma}, \quad \text{for any } s \in [-1, 1].$$

Proof. In light of (A.50), we have for all $s \in [-1, 1]$,

$$\left| \sum_{k \geq 0} \varphi^{(k)}(s) \frac{1}{k!} (z - s)^k \right| \leq \sum_{k \geq 0} M(2\rho)^{-k} |z - s|^k.$$

This entails that for all $s \in [-1, 1]$ and for all $z \in B(s, \rho)$, we have the following estimation

$$(A.52) \quad \left| \sum_{k \geq 0} \varphi^{(k)}(s) \frac{1}{k!} (z - s)^k \right| \leq M \sum_{k \geq 0} (2\rho)^{-k} \rho^k \leq 2M,$$

which implies that φ can be extended to an holomorphic function in $D_\rho = \cup B(s, \rho)$ for $-1 \leq s \leq 1$. We need first to construct a specific open interval in $[-\frac{1}{5}, \frac{1}{5}]$ to apply Lemma A.1. To this end, we notice that

$$(A.53) \quad [-1, 1] \subset \bigcup_{1 \leq j \leq n_0} I_j = \bigcup_{1 \leq j \leq n_0} \left[s_j - \frac{\rho}{5}, s_j + \frac{\rho}{5} \right],$$

where we have putted $s_j = -1 + (2j - 1)\rho/5$, $5/\rho \leq n_0 \leq 5/\rho + 1/2$ and assumed that $I_j \cap I_{j'} = \emptyset$, for all $j, j' = 1, \dots, n_0$, $j \neq j'$. Therefore, the open interval I can be written as the meeting of $(I_j \cap I)$, for $1 \leq j \leq n_0$ where

$$(I_j \cap I) \bigcap_{j \neq j'} (I_{j'} \cap I) = \emptyset, \quad \text{for } j, j' = 1, \dots, n_0.$$

Now, we fix $j_0 \in \{1, \dots, n_0\}$ such that $|I_{j_0} \cap I| = \max_{1 \leq j \leq n_0} |I_j \cap I|$. We define $J_{s_{j_0}, \rho} = \frac{1}{\rho}(I_{j_0} \cap I - s_{j_0})$.

In light of (A.53), we deduce that $J_{s_{j_0}, \rho}$ is an open interval of $[-\frac{1}{5}, \frac{1}{5}]$. Next, we consider the function g defined on $D(0, 1)$ as follows

$$g(z) = \frac{\varphi(s_{j_0} + \rho z)}{2M}.$$

The estimate (A.52) entails that $|g(z)| \leq 1$ for $|z| \leq 1$. Bearing in mind that the function g is holomorphic in the unit disc, we deduce from Lemma A.1 the existence of two constants $N = N(|I|)$ and $\gamma = \gamma(|I|)$ such that the following estimate holds

$$\|g\|_{L^\infty(B(0, 1/2))} \leq N \|g\|_{L^\infty(J_{s_{j_0}, \rho})}^\gamma \leq N (2M)^{-\gamma} \|\varphi\|_{L^\infty(I_{j_0} \cap I)}.$$

This combined with the fact that $\|g\|_{L^\infty(B(0, 1/2))} = (2M)^{-1} \|\varphi\|_{L^\infty(B(s_{j_0}, \rho/2))}$ yield the following result

$$(A.54) \quad \|\varphi\|_{L^\infty(B(s_{j_0}, \rho/2))} \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma}.$$

Now, we aim to extend this result to the interval $[-1, 1]$. To this end, let $r > 0$, satisfying

$$(A.55) \quad \frac{\rho}{2} \leq r \leq 2r \leq \rho.$$

Let $(a_j)_{j \geq 1} = (s_j)_{j \geq 1}$ be a sequence such that $[-1, 1] \subset \bigcup_{1 \leq j \leq n_0} B(a_j, 2r)$ and satisfying

$$(A.56) \quad \begin{cases} B(a_{j+1}, r) \subset B(a_j, 2r) & \text{for } j \in \{j_0, \dots, n_0\} \\ B(a_{j-1}, r) \subset B(a_j, 2r) & \text{for } j \in \{1, \dots, j_0\}. \end{cases}$$

In view of Hamdamard's three-circle theorem, using (A.54) and (A.55) we get

$$(A.57) \quad \|\varphi\|_{L^\infty(B(a_{j_0}, 2r))} \leq \|\varphi\|_{L^\infty(B(a_{j_0}, \frac{\rho}{2}))}^\theta \|\varphi\|_{L^\infty(B(a_{j_0}, \rho))}^{1-\theta} \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma},$$

where $\theta = \frac{\log \rho/2r}{\log 2}$. Then, using the fact that $B(a_{j+1}, r) \subset B(a_j, 2r)$ for $j \in \{j_0, \dots, n_0\}$, we deduce

$$\|\varphi\|_{L^\infty(B(a_{j_0+1}, r))} \leq \|\varphi\|_{L^\infty(B(a_{j_0}, 2r))} \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma}.$$

From this and Hadamard's three-circle theorem, we obtain

$$\|\varphi\|_{L^\infty(B(a_{j_0+1}, 2r))} \leq \|\varphi\|_{L^\infty(B(a_{j_0+1}, r))}^{\theta'} \|\varphi\|_{L^\infty(B(a_{j_0+1}, \rho))}^{1-\theta'} \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma},$$

where $\theta' = \frac{\log \rho/2r}{\log \rho/r}$. So, from (A.56) and a repeated application of Hadamard's three circle theorem, we get

$$\|\varphi\|_{L^\infty(B(a_j, 2r))} \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma}, \quad j \in \{j_0 + 2, \dots, n_0\}.$$

By a similar way, we prove that

$$\|\varphi\|_{L^\infty(B(a_j, 2r))} \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma}, \quad j \in \{1, \dots, j_0\}.$$

As a consequence, we obtain

$$\|\varphi\|_{L^\infty([-1, 1])} \leq \sum_{j=1}^{n_0} \|\varphi\|_{L^\infty(B(a_j, 2r))} \leq N \|\varphi\|_{L^\infty(I)}^\gamma M^{1-\gamma}.$$

This completes the proof of the Lemma. \square

A.1. Proof of Lemma 3.4. Notice first that there exists a sequence of open intervals $(I_j)_j$ such that

$$E = I_1 \times \dots \times I_j \times \dots \times I_d \subset \mathcal{O} \subset B(0, 1).$$

Let $x = (x_1, x_2, \dots, x_d)$ be fixed in $B(0, 1)$. We consider the analytic function φ_j defined as follows

$$(A.58) \quad \varphi_j(s) = F(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_d), \quad s \in [-1, 1].$$

Assume that there exist positive constants M and ρ such that

$$|\varphi_j(s)^{(k)}| \leq \frac{Mk!}{(2\rho)^k}, \quad s \in [-1, 1].$$

Then, in view of lemma A.2, we conclude the existence of $N = N(\rho, |I_j|)$ and $\gamma = \gamma(\rho, |I|)$ such that we have

$$|\varphi_j(s)| \leq N \|\varphi_j\|_{L^\infty(I_j)}^{\gamma_j} M^{1-\gamma_j}, \quad s \in [-1, 1],$$

This and (A.58) yield

$$(A.59) \quad |F(x)| \leq N_j \sup_{x_j \in I_j} |F(x)|^{\gamma_j} M^{1-\gamma_j}.$$

Therefore, by iterating (A.59), we get

$$|F(x)| \leq N_1 N_2^{\gamma_1} \dots N_d^{\gamma_1 \dots \gamma_{d-1}} \sup_{x \in E} |F(x)|^{\gamma_1 \dots \gamma_d} M^{1-\gamma_1 \dots \gamma_d}.$$

This completes the of the lemma.

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